Math 19. Lecture 25
Summary of Advection/Diffusion

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1 Advection and Diffusion

The advection equation is given by

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + f(u). \tag{1}$$

The diffusion equation is given by

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + f(u). \tag{2}$$

- c and μ are determined experimentally.
- The advection equation is used when the particle motion is due to the motion of the ambient fluid.
- The diffusion model is used when the particles move randomly.
- Sometimes advection and diffusion are both at work:

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x} + f(u).$$

- For either the advection or diffusion equation, there may be many solutions. The solution for a particular problem depends heavily on initial and boundary conditions.

2 Equilibrium Solutions

An equilibrium solution to (1) or (2), is a solution $u = u(x,t)$ that is independent of time. Thus, $u = u_e(x)$ must either obey

$$-c\frac{du_e}{dx} + f(u_e) = 0$$

or

$$\mu\frac{d^2u_e}{dx^2} + f(u_e) = 0,$$

plus the relevant boundary conditions.

3 Stability

Let u_e be an equilibrium solution to

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} + f(u)$$
$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,L) = 0.$$

The solution $u_e(x)$ is a *linearly stable* solution to

$$\mu\frac{d^2u_e}{dx^2} + f(u_e) = 0$$
$$\frac{d}{dx}u_e(0) = \frac{d}{dx}u_e(L) = 0$$

if and only if there is *no* pair (g, λ) , where $g(x)$ is some function that is *not* identically zero for $0 \leq x \leq L$, where $\lambda \in \mathbb{R}$, and where the following constraints are satisfied.

- $\lambda \geq 0$
- $\lambda g = \mu\frac{d^2}{dx^2} + z(x)g$
- $\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=L} = 0$

A solution is unstable if there is even one such pair (g, λ) that obeys the above conditions.

4 An Example for Advection

The equation

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - \frac{u(1-u)}{100}$$

has a time independent solution

$$u_e(x) = \frac{1}{e^{x/100}+1}$$

that models bacteria concentrations in a river that is downstream from a sewage treatment plant, where x is the distance downstream from the plant.

5 An Example for Diffusion

The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2$$

might model the fish concentration in a square lake that is stocked at both ends. Consider a rectangular lake of constant width and depth. Suppose that fish are dumped into the lake at either end such that the concentration of the fish at the ends of the lake are three fish per 100 m^3. If $-\pi/2 \leq x \leq \pi/2$ and

$$u(t, -\pi/2) = u(t, \pi/2) = 3, \tag{3}$$

then we have an equilibrium solution

$$u_e(x) = \frac{3}{\cos x + 1}.$$

In order to verify stability, we must check solutions to the equation

$$\lambda g = \frac{d^2 g}{dx^2} + \frac{\cos x - 5}{1 + \cos x} g, \tag{4}$$

with

$$g(-\pi/2) = g(\pi/2) = 0.$$

If g satisfies these conditions, then

$$u(t, x) = e^{\lambda t} g(x) + u_e(x)$$

will satisfy (3). Given the complexity of u_e, solving (4) may be difficult or even impossible.

6 The Maximum Principle

We can use the *Maximum Principle* to analyze (4). Here is the idea. Assume that you have found a pair (λ, g) satisfying (4) such that g is not identically zero. We will argue that $\lambda < 0$. If this is the case, then we have a stable solution.

- First, g must have a maximum and a minimum on $[-\pi/2, \pi/2]$. Furthermore, the maximum and the minimum cannot be the same. If they were, then g would have to be a constant function. Since we know g at the endpoints, g must be identically zero. In this case, which we assumed could not be the case.

- Suppose that $g > 0$ at its maximum.[1] Then g must be concave down at this point and $d^2g/dx^2 \leq 0$ here. Since $g > 0$ and $-1 \leq \cos x \leq 1$, we know that
$$\left(\frac{\cos x - 5}{1 + \cos x}\right) g < 0.$$
Thus, $\lambda g < 0$, which tells us that $\lambda < 0$.

- On the other hand, suppose that $g \leq 0$ and has no positive maximum. Then g must have a negative minimum, and $d^2g/dx^2 \geq 0$ at the minimum. Since $g < 0$ and
$$\frac{\cos x - 5}{1 + \cos x} < 0,$$
we know that the right side of (4) is positive. Therefore, λ must be less than zero in order for the left side to be positive.

Our argument depends heavily on the boundary conditions and $f(u)$.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 20.

[1] We divided this into cases: $g > 0$ and $g \leq 0$.