Math 19. Lecture 19
Separation of Variables (I)

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1 Solutions to the ODE $B'' = cB$

We will divide the solution of $B'' = cB$, depending on the sign of c.

- *Case 1*: $c = 0$. If $B'' = 0$, then B must be a linear function. Therefore,

$$B(x) = \alpha + \beta x.$$

- *Case 2*: $c > 0$. If $B'' = cB$ and $c > 0$, then we can let $\lambda^2 = c$. Thus, we must solve the equation

$$B'' - \lambda^2 B = 0.$$

Let us assume that our solution has the form $B(x) = e^{rx}$. Then

$$\frac{d^2}{dx^2}B(x) - \lambda^2 B(x) = r^2 e^{rx} - \lambda^2 e^{rx} = (r^2 - \lambda^2)e^{rx}.$$

Since e^{rx} is never zero,

$$r^2 - \lambda^2 = (r - \lambda)(r + \lambda) = 0,$$

$r = \pm\lambda$. Thus, we have solutions

$$B(x) = e^{\lambda x} \text{ and } B(x) = e^{-\lambda x}.$$

Using the Principle of Superposition, our solution is

$$B(x) = \alpha e^{\lambda x} + \beta e^{-\lambda x}.$$

- *Case 3*: $c < 0$. If $B'' = cB$ and $c < 0$, then we can let $-\lambda^2 = c$. Thus, we must solve the equation

$$B'' + \lambda^2 B = 0.$$

It is easy to verify that

$$B(x) = \cos \lambda x \text{ and } B(x) = \sin \lambda x.$$

Using the Principle of Superposition, our solution is

$$B(x) = \alpha \cos \lambda x + \beta \sin \lambda x.$$

2 Uniqueness of Solutions

We must still determine that these solutions are unique. If we introduce a new variable, $P = B'(x)$, we can rewrite the equation $B'' = cB$ as a linear system of ordinary differential equations,

$$\begin{aligned} \frac{dB}{dx} &= P \\ \frac{dP}{dx} &= cB. \end{aligned}$$

However, any 2×2 linear system of ODEs is completely determined by the value of P and B at $x = 0$.

- *Case 1*: If $c = 0$, then we have solution $B(x) = \alpha + \beta x$ and $P(x) = \beta$. Thus, $P(0) = \beta$ and $B(0) = \alpha$.

- *Case 2*: If $\lambda^2 = c > 0$, then

$$\begin{aligned} B(x) &= \alpha e^{\lambda x} + \beta e^{-\lambda x}, \\ P(x) &= \alpha \lambda e^{\lambda x} - \beta \lambda e^{-\lambda x}. \end{aligned}$$

Therefore,

$$\begin{aligned} B(0) &= \alpha + \beta, \\ P(0) &= \alpha\lambda - \beta\lambda. \end{aligned}$$

- *Case 3*: If $-\lambda^2 = c < 0$, then

$$\begin{aligned} B(x) &= \alpha \cos \lambda x + \beta \sin \lambda x \\ P(x) &= -\alpha\lambda \sin \lambda x + \beta\lambda \cos \lambda x. \end{aligned}$$

  Therefore,

$$\begin{aligned} B(0) &= \alpha \\ P(0) &= \beta\lambda. \end{aligned}$$

3 Modeling the Density of Protein

It is known that the concentration of certain proteins at any cell in an embryo determines whether or not a particular gene is expressed in that cell. We will consider a cell model of an embryo where

$$u(t, x, y)$$

is the density of protein at time t and position (x, y). We will consider our embryo to be square, $[0, L] \times [0, L]$, where Protein is produced along the left-hand edge according to

$$u(t, 0, y) = \sin\left(\frac{\pi y}{L}\right).$$

Observe that this function is zero at $(0, 0)$ and $(0, L)$. Assume also that

$$\begin{aligned} u(t, x, 0) &= 0 \\ u(t, x, L) &= 0 \\ u(t, L, y) &= 0. \end{aligned}$$

The protein will diffuse according to the equation

$$\frac{\partial u}{\partial t} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - ru.$$

4 Equilibrium Solutions

If there is no time dependence, then

$$\frac{\partial u}{\partial t} = 0.$$

In this case

$$\frac{\partial u}{\partial t} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - ru$$

becomes either

- *Helmholtz's Equation*:

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - ru = 0$$

- *Laplace's Equation*:

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 17.