Math 19. Lecture 17
Advection and Diffusion—Key Properties

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1 Existence and Uniqueness of Solutions

Recall the *advection equation*

$$u_t = -cu_x + ru$$

and the *diffusion equation*

$$u_t = \mu u_{xx} + ru.$$

These equations are completely predictive. If we specify an *initial condition*

$$u(0, x) = f(x).$$

the both the advection and diffusion equations determine $u(t, x)$ for all $t \geq 0$.

2 The Idea of the Proof

Since we know that $u(0, t) = f(x)$, the equation tells us what $u_t(0, x)$ for all x. This tells us what $u(\Delta t, x)$ is for small Δt and all x. Put $u(\Delta t, x)$ into the left side of the PDE to find out what u is at $t = \Delta t$ for any x. This tells us what $u(2\Delta t, x)$ is for small Δt and all x. Now continue ...

3 Solutions

We already know that the advection equation has solutions of the form

$$u(t,x) = e^{rt}u(0, x - ct),$$

where $u(0, x - ct) = f(x - ct)$. The fundamental solutions of the diffusion equation are a bit more complicated,

$$u(t,x) = \frac{1}{\sqrt{4\pi\mu t}} e^{rt} \int_{-\infty}^{\infty} u(0,s) e^{-(x-s)^2/4\mu t} \, ds.$$

This solution is a bit too complicated to be useful.

4 The Superposition Principle

For any linear PDE, the sum of two solutions is a solution, and multiple of a solution is a solution. That is, if u_1 and u_2 are solutions to a linear PDE, then

$$\alpha u_1 + \beta u_2 \tag{1}$$

is a solution. An expression of the form (1) is called a *linear combination* of u_1 and u_2. *Show this for*

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + ru.$$

5 Building Solutions

The Principle of Superposition allows us to build complicated solutions out of simple solutions. For example, if

$$\begin{aligned} u_1(t,x) &= \frac{1}{t^{1/2}} e^{-x^2/4\mu t} \\ u_2(t,x) &= \frac{1}{t^{1/2}} e^{-(x-1)^2/4\mu t} \\ u_3(t,x) &= \frac{1}{t^{1/2}} e^{-(x+1)^2/4\mu t} \end{aligned}$$

are solutions to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + ru,$$

then

$$u(t,x) = 3u_1(t,x) - 2u_2(t,x) + 5u_3(t,x)$$

is also a solution.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 15.