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ON THE SEMILOCAL CONVERGENCE OF MULTI-POINT WEIERSTRASS TYPE ROOT-FINDING METHODS FOR SIMULTANEOUS APPROXIMATION OF POLYNOMIAL ZEROS

PETKO D. PROINOV AND MILENA D. PETKOVA

Abstract

In this talk, we discuss the convergence of a family of multi-point iterative methods for approximating all zeros of a polynomial simultaneously. This family was introduced by Kyurkchiev and Ivanov [2] and it is based on the famous Weierstrass root-finding method. Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial of degree $n \geq 2$ with coefficients in a valued field $\mathbb{K}$. Recall that the Weierstrass method is defined in $\mathbb{K}^n$ as follows

$$x^{(k+1)} = x^{(k)} - W_f(x^{(k)}), \qquad k = 0, 1, 2, \ldots,$$

where the Weierstrass correction $W_f \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by

$$W_f(x) = (W_1(x), \ldots, W_n(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i}(x_i - x_j)} .$$

By analogy, we can define the iterative function $F \colon D \subset \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$ by

$$F(x, y) = (F_1(x, y), \ldots, F_n(x, y)) \quad \text{with} \quad F_i(x, y) = x_i - \frac{f(x_i)}{a_0 \prod_{j \neq i}(x_i - y_j)} .$$

We define a sequence $(T^{(N)})_{N=1}^{\infty}$ of iteration functions $T^{(N)} \colon D_N \subset \underbrace{\mathbb{K}^n \times \ldots \times \mathbb{K}^n}_{N+1} \to \mathbb{K}^n$ recursively by setting $T^{(1)}(x) = F(x, y)$ and $T^{(N)}(x, y, \ldots, z) = F(x, T^{(N-1)}(y, \ldots, z))$.

Now, for given $N \in \mathbb{N}$ and initial approximations $x^{(0)}$, $x^{(1)}$, $\ldots$, $x^{(N)}$ in $\mathbb{K}^n$, the N-th iterative method of Kyurkchiev-Ivanov's family can be defined by the following iteration

$$x^{(k+1)} = T^{(N)}\left(x^{(k)}, x^{(k-1)}, \ldots, x^{(k-N)}\right), \qquad k = N, N+1, N+2, \ldots. \tag{1}$$

The aim of this talk is to present a convergence theorem for the multi-point Weierstrass type methods (1) with computationally verifiable initial conditions. In the case $N = 1$,

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this theorem improves our previous result given in [4]. The theorem is obtained by a new approach to the iteration functions of multi-point simultaneous methods and by using an initial condition conversion theorem of [3].

The following theorem is a consequence of the main result of our study.

Theorem 1. *Let the initial approximations* $x^{(0)}, x^{(1)}, \ldots, x^{(N)} \in \mathbb{K}^n$ *satisfy the condition*

$$\max_{0 \le k \le N} \frac{\left\| W_f(x^k) \right\|_\infty}{\delta(x^k)} < \frac{(\sqrt[n-1]{2} - 1)(3\sqrt[n-1]{2} - 2)}{(4\sqrt[n-1]{2} - 3)((n+2)\sqrt[n-1]{2} - n - 1)},$$

where the function $\delta \colon \mathbb{K}^n \to \mathbb{R}_+$ *is defined by* $\delta(x) = \min_{i \neq j} |x_i - x_j|$. *Then* f *has* n *simple zeros in* $\mathbb{K}$ *and the multi-point Weierstrass iteration* (1) *is well defined and converges* (*with respect to an appropriate metric*) *to the zeros of* f *with order of convergence* $r = r(N)$, *where* r *is the unique positive solution of the equation* $1 + t + \ldots + t^N = t^{N+1}$.

Keywords: multipoint iterative methods, polynomial zeros, simultaneous methods, local convergence, semilocal convergence, error estimates.

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Petko D. Proinov,
Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski,
24 Tzar Asen, 4000 Plovdiv, Bulgaria.
`proinov@uni-plovdiv.bg`

Milena D. Petkova,
Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski,
24 Tzar Asen, 4000 Plovdiv, Bulgaria.
`milenapetkova.pu@gmail.com`