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EXPLICIT ALGEBRAIC SOLUTION TO ZOLOTAREV'S »FIRST PROBLEM« FOR POLYNOMIALS OF DEGREE $n \in \{6, 7, 8\}$

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Abstract

E. I. Zolotarev's so-called First Problem (ZFP) of 1877 ([1], [2], [4], [8], [16]) extends P. L. Chebyshev's approximation problem of 1854 [3] and calls for a polynomial $P^*_{n,s}$ of degree n with *two* fixed leading coefficients, 1 and $-ns$ (where $s \in \mathbb{R}$ is given), i.e. $P^*_{n,s}(x) = \sum_{k=0}^{n-2} a^*_{k,n}(s)x^k + (-ns)x^{n-1} + x^n$, which deviates least from the zero-function on $[-1, 1]$ (in the uniform norm). It suffices to consider $s \geq 0$, and we focus on the non-trivial cases $s > tan^2(\frac{\pi}{2n})$ for which the resulting $P^*_{n,s}$ is called a proper or hard-core monic Zolotarev polynomial. It alternates n times on $[-1, 1]$ and 2 times on some $[\alpha, \beta]$, including the endpoints of these intervals as alternation points, where $1 < \alpha < \beta$. Zolotarev himself provided a transcendental solution for all $n \geq 2$ in terms of elliptic functions [16]. The provision of an algebraic solution to ZFP has been vibrant from the outset, and was restated in [5] as an open problem to be solved on a computer, for $n \geq 6$. The cases $2 \leq n < 6$ are settled [11].

We provide, accompanied by an example, an explicit algebraic solution to ZFP for polynomials of degree $n \in N := \{6, 7, 8\}$ by determining the optimal coefficients $a^*_{k,n}(s)$ of the proper monic Zolotarev polynomial $P^*_{n,s}$ in four traceable steps. The first two of these are:

1. To express, for each $n \in N$, tentative coefficients $a_{k,n}(\alpha, \beta)$ of $P^*_{n,s}$ as integer rational functions of α and β, where $0 \leq k \leq n - 2$.
2. To calculate, for each $n \in N$, a pair of dedicated polynomials $F_{m(n),s}$ and $G_{m(n),s}$ of degree $m(6) = 8$, resp. $m(7) = 12$, resp. $m(8) = 16$, whose coefficients $b_{l,m(n)}(s)$ and $c_{l,m(n)}(s)$ are integer polynomial functions of s, where $0 \leq l \leq m(n)$.

In 2004 A. Shadrin [13] remarked: *Recently, the interest in an explicit algebraic solution of ZFP was revived in the papers [7], [9], and [14], but it is only Malyshev who*

demonstrates how his theory can be applied to some explicit constructions for particular n. Actually, V.A. Malyshev [7] calculated the polynomials $F_{m(n),s}$ and $G_{m(n),s}$ for $2 \leq n < 6$ and predicted correctly, as we have verified in step 2, their degrees for $n \in N$.

Our approach to ZFP for low-degree polynomials ought to be contrasted with the approach via symbolic computation by D. Lazard [6] of 2006, where a solution for $6 \leq n \leq 12$ is claimed, but no explicit example is shared. In 2007 K. Schiefermayr [12] presented a methodical algebraic approach to ZFP based on results of [10], [15], without however going into details for $n \in N$.

We point out that the here introduced general coefficients $a_{k,n}(\alpha, \beta)$, $b_{l,m(n)}(s)$ and $c_{l,m(n)}(s)$, which facilitate the determination of the sought-for $P^*_{n,s}$ (where $n \in N$), do not appear in any of the references above. Once these coefficients are stored, the construction of $P^*_{n,s}$ for any given $n \in N$ and any given $s > tan^2(\frac{\pi}{2n})$ becomes straightforward. Remark: This Abstract is adapted from our poster, see Poster Session. Our participation in the Conference will be online only.

Keywords: algebraic solution, approximation, first problem, least deviation, low-degree polynomial, poster, Zolotarev.

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